1.1 INTRODUCTION

Mathematical analysis studies concepts related in some way to real numbers, so we begin our study of analysis with a discussion of the real-number system.

Several methods are used to introduce real numbers. One method starts with the positive integers 1, 2, 3, ... as undefined concepts and uses them to build a larger system, the positive rational numbers (quotients of positive integers), their negatives, and zero. The rational numbers, in turn, are then used to construct the irrational numbers, real numbers like $-\sqrt{2}$ and iv which are not rational. The rational and irrational numbers together constitute the real-number system.

Although these matters are an important part of the foundations of mathematics, they will not be described in detail here. As a matter of fact, in most phases of analysis it is only the properties of real numbers that concern us, rather than the methods used to construct them. Therefore, we shall take the real numbers themselves as undefined objects satisfying certain axioms from which further properties will be derived. Since the reader is probably familiar with most of the properties of real numbers discussed in the next few pages, the presentation will be rather brief. Its purpose is to review the important features and persuade the reader that, if it were necessary to do so, all the properties could be traced back to the axioms. More detailed treatments can be found in the references at the end of this chapter.

For convenience we use some elementary set notation and terminology. Let S denote a set (a collection of objects). The notation $x \varepsilon S$ means that the object x is in the set S, and we write $x \varepsilon S$ to indicate that x is not in S.

A set S is said to be a subset of T, and we write $S \leq T$, if every object in S is also in T. A set is called nonempty if it contains at least one object.

We assume there exists a nonempty set R of objects, called real numbers, which satisfy the ten axioms listed below. The axioms fall in a natural way into three groups which we refer to as the field axioms, the order axioms, and the completeness axiom (also called the least-upper-bound axiom or the axiom of continuity).

1.2 THE FIELD AXIOMS

Along with the-set R of real numbers we assume the existence of two operations, called addition and multiplication, such that for every pair of real numbers x and y the sum $x + y$ and the product xy are real numbers uniquely determined by x and y satisfying the following axioms. (In the axioms that appear below, x, y, z represent arbitrary real numbers unless something is said to the contrary.)

Axiom 1. $x + y = y + x$, $xy = yx$ (commutative laws).
Axiom 2. $x + (y + z) = (x + y) + z$, $x(yz)$ _ $(xy)z$ (associative laws).
Axiom 3. $x(y + z) = xy + xz$ (distributive law).
Axiom 4. Given any two real numbers x and y, there exists a real number z such that $x + z = y$. This z is denoted by $y - x$; the number $x - x$ is denoted by 0. (It can be proved that 0 is independent of x.) We write $-x$ for $0 - x$ and call $-x$ the negative of x.
Axiom 5. There exists at least one real number $x \neq 0$. If x and y are two real numbers with $x \neq 0$, then there exists a real number z such that $xz = y$. This z is denoted by y/x; the number x/x is denoted by 1 and can be shown to be independent of x. We write x^{-1} for $1/x$ if $x \neq 0$ and call x^{-1} the reciprocal of x.

From these axioms all the usual laws of arithmetic can be derived; for example, $-(-x) = x$, $(x^{-1})^{-1} = x$, $-(x - y) = y - x$, $x - y = x + (-y)$, etc. (For a more detailed explanation, see Reference 1.1.)

1.3 THE ORDER AXIOMS

We also assume the existence of a relation < which establishes an ordering among the real numbers and which satisfies the following axioms:

Axiom 6. Exactly one of the relations $x = y$, $x < y$, $x > y$ holds.
NOTE. $x > y$ means the same as $y < x$.
Axiom 7. If $x < y$, then for every z we have $x + z < y + z$.
Axiom 8. If $x > 0$ and $y > 0$, then $xy > 0$.
Axiom 9. If $x > y$ and $y > z$, then $x > z$.

NOTE. A real number x is called positive if $x > 0$, and negative if $x < 0$. We denote by $\mathbf{R}^+$ the set of all positive real numbers, and by $\mathbf{R}^-$ the set of all negative real numbers.

From these axioms we can derive the usual rules for operating with inequalities. For example, if we have $x < y$, then $xz < yz$ if z is positive, whereas $xz > yz$ if z is negative. Also, if $x > y$ and $z > w$ where both y and w are positive, then $xz > yw$. (For a complete discussion of these rules see Reference 1.1.)

NOTE. The symbolism $x \leq y$ is used as an abbreviation for the statement:

"$x < y$ or $x = y$"

Thus, we have $2 \leq 3$ since $2 < 3$; and $2 \leq 2$ since $2 = 2$. The symbol $\geq$ is similarly used. A real number x is called nonnegative if $x \geq 0$. A pair of simultaneous inequalities such as $x < y$, $y < z$ is usually written more briefly as $x < y < z$.

The following theorem, which is a simple consequence of the foregoing axioms, is often used in proofs in analysis.

Theorem 1.1. Given real numbers a and b such that

$$a < b + \varepsilon \text{ for every } e > 0. \qquad (1)$$

Then $a \leq b$.

Proof. If $b < a$, then inequality (1) is violated for $e = (a - b)/2$ because

$$b + \varepsilon = b + \frac{a-b}{2} = \frac{a+b}{2} < \frac{a+a}{2} = a$$

Therefore, by Axiom 6 we must have $a \leq b$.

Axiom 10, the completeness axiom, will be described in Section 1.11.

RATIONAL NUMBERS :

Quotients of integers a/b (where b≠0) are called **RATIONAL NUMBERS.** For Example1/2, -7/5 and 6 Rational Numbers. The set of rational numbers which we denote by Q, contains Z as a subset.

If a and b are Rational, their average (a+b/2) is also Rational and lies between a and b. Therefore between any two rational numbers there are infinitely many rational numbers which implies that if we are given a certain rational number we cannot speak of the **"NEXT LARGEST"** rational number.

IRRATIONAL NUMBERS :

Real Numbers that are not Rational are called **"IRRATIONAL"** . For Example the numbers $\sqrt{2}$, e, π, and e^x are Irrational.

Theorem 1.10 :

If n is a positive integer which is not a perfect square, then $\sqrt{n}$ is irrational.

Proof :

Case (i) :

Suppose first that n contains no square factor > 1. ....(1)

Assume $\sqrt{n}$ is rational.

$\sqrt{n} = a/b$ (where a and b are integers having no factor in common)

$n = a^2/b^2$ ....(2)

$nb^2 = a^2$ ....(3)

since the left side of this equation is a multiple of n. So too is a^2.

The n^{th} term has the sign of the fisrt neglected term and is less in absolute value.

$S_n = \sum = 0\ (-1)^k/k!$

We have the inequality.

$$0 < e^{-1} - S_{2k-1} < 1/(2k)!$$

$0 < (e^{-1} - S_{2k-1})\ (2k-1) < 1/2k \leq 1/2$ **(for any integer $k \geq 1$)** **..........(1)**

N0w , (2k-1)! $(e^{-1} - S_{2k-1})$ is always integer.

If e^{-1} were rational. Then we could choose k so large that

$$k > (2k-1)!\ e^{-1}$$

(1) The different of the two integers would be a number lies between o and 1/e which is impossible.

Thus e^{-1} cannot be rational and hence e cannot be rational.

$\therefore$ e is irrational.

Hence the proof.

1.10. UPPER BOUNDS, MAXIMUM ELEMENT

Irrational number arise in algebra when we try to solve certain quadratic equations. For example, it is desirable to have a real number x such that $x^2 = 2$.From the nine axioms listed above we cannot prove that such an x exists in R because these nine axioms are also satisfied by Q and we have shown that there is no rational number whose square is Z. The completeness axiom allows us to introduce irrational numbers in the real number system, and it gives the real-number system a property of continuity that is fundamental to many theorems in analysis.

Before we describe the completeness axiom, it is convenient to introduce additional terminology and notation.

Definition 1.12. Let S be a set of real numbers. If there is real number b such that $x \leq b$ for every x in S, then b is called an upper bound for S and we say that S is bounded above by b.

We say an upper bound because every number greater than b will also be an upper bound. If an upper bound b is also a member of S, then b is called the largest member or the maximum element of S. There can be at most one such b. If it exists, we write

$$b = max\, S.$$

A set with no upper bound is said to be unbounded above.

Definitions of the terms lower bound, bounded below, smallest member (or minimum element) can be similarly formulated. If S has a minimum element we denote it by min S.

Examples

1.The set $R^+ = (0, +\infty)$ is unbounded above. It has no upper bounds and no maximum element. It is bounded below by 0 but has no minimum element.

2.The closed interval $S = [0,1]$ is bounded above by 1 and is bounded below by 0. In fact, max S=1 and min S=0.

3.The half open interval $S = [0,1)$ is bounded above by 1 but it has no maximum element. Its minimum element 0.

LEAST UPPER BOUND (OR) SUPREMUM

DEFINITION 1.13

Let S be a set of real numbers bounded above. A real number b is called a least upper bound for S if it has the following two properties:

a) b is an upper bound for S.

b) No number less than b is an upper bound for S.

EXAMPLES:

If $S = [0,1]$ the maximum element 1 is also a least upper bound for S. If $[0,1]$ the number 1 is a least upper bound for S, even though S has no maximum element.

It is an easy exercise to prove that a set cannot have two different least upper bounds. Therefore, if there is least upper bound for S, there is only one and we can speak of the least upper bound.

It is common practice to refer to the least upper bound of a set by the more concise term supreme, abbreviated sup. We shall adopt this convention and write

$$b = \sup S$$

to indicate that b is the supremum of S. If S has a maximum element, then max $S = \sup S$.

The greatest lower bound, or infimum of S, denoted by inf S, is defined in an analogous fashion.

THE COMPLETENESS AXIOM

Our final axiom for the real number system involves the notion of supremum.

Axiom 10. Every nonempty set S of real numbers which is bounded above has a supremum; that is, there is a real number b such that $b = \sup S$.

As a consequence of this axiom it follows that every nonempty of real numbers which is bounded below has an infimum.

TRINGLE INEQUALITY

If x is any real number, the absolute value of x_1 denoted by (x) is defined as follows.

$$|x| = \{ x \text{if } x \geq 0 \text{-}x \text{if } x \leq 0$$

A fundamental inequality concerning absolute values is given in the following.

Theorem 1.21

If $a \geq 0$ then we have the in equality $|x| \leq a$ if and only if $-a \leq x \leq a$.

Proof:

$|x|$ we have inequality

$-|x| \leq x \leq x$

$X = |x|$ or $x = -|x|$

Assume $|x| \leq a$

$-|x| \geq -a$

$-a \leq -x \leq x \leq |x| \leq a$

$-a \leq x \leq a$

They if $x \geq 0$

$|x| = x \leq a$

If $x < 0$

$|x| = -x \leq a.$

Theorem 1.22:

For arbitrary real x and y we have $|X+Y| \leq |X|+|Y|$

Proof:

We have

$-|X|\leq|X|\leq X$

$-|Y|\leq Y\leq|Y|$

Addiction

$(|X|+|Y|)\leq X+Y\leq|X|+|Y|$

$|X+Y|\leq|X|+|Y|$

But $X=A-C$, $Y=C-B$

$X+Y=A-C=AB=A-B$

$|a-b|\leq|a-c|+|c-b|$

$|z|\geq|x+y|-|y|$

$X=A=B$, $y=-b$

$|A+B|\geq|A|-|B|$

$|A+B|\geq|B|-|A|=-(|A|-|B|)$

$|A+B|\geq(|A|-|B|$

By induction we can also prove the generdigations

$|x_1+x_2+......+x_n|\leq|x_1|-|x_2|.....+|x_n|$

$|x_1+x_2+...... x_n| \geq |x_1|-|x_2|......-|x_n|$

Theorem: 1.23:

If $a_1,.......,a_n$ and $b_1,........,b_n$ are arbitrary real numbers, we have

$$(k=1na_k\, b_k)^2 \leq (k=1na2_k)\ (k=1nb^2_k)$$

Moreover of come $a_1 \neq 0$ equality holds if and only if there is a real x such that $a_k x+a_k=0$ for each k=1, 2, ... n.

Proof:

A sum of squares can never be negative hence we have.

$$k=1n(a_k\, x + b_k)^2 \geq 0$$

For every real x, with equality if and only if reach term is zero the inequality can be written in the form

$Ax^{2+}2bx+c \geq 0$

Where

$$A=k=1na^2_k \quad B=k=1na_k\, b_k \quad C=k=1nb^2_k$$